Bridging between eddy-viscosity-type and second-order turbulence models through a two-scale turbulence theory

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A gap between eddy-viscosity-type and second-order models is bridged using the results of a two-scale direct-interaction approximation developed for the study of turbulent shear flows. This work provides a method for incorporating the findings from models of eddy-viscosity-type into second-order models, and vice versa. Specifically, the effect of helicity controlling energy-cascade processes is incorporated into a second-order model. Then, a higher-order eddy-viscosity-type expression for the Reynolds stress is derived through the application of an iterative approximation to the second-order model. The latter result is tested in a turbulent rotating channel flow and its usefulness is confirmed. Effects of flow trajectory are also discussed in the context of the effect of an adverse pressure gradient on the isotropic eddy viscosity.

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I. INTRODUCTION

Turbulence models based on one-point quantities such as the mean velocity and the turbulent energy have been studied extensively for several reasons. One reason is that all the scales in a turbulent motion appearing in important engineering and scientific phenomena cannot be resolved simultaneously using the largest computer now available. Another reason is related to the abstraction of intrinsic properties of a turbulent motion. Even if all the scales in a turbulent flow can be resolved in a computer simulation, it is necessary to abstract compactly the characteristics of the flow from a formidable amount of numerical data for a clear understanding of turbulence mechanism. In the one-point turbulence modeling, attention is focused on the mean-field and energetic parts of fluctuations. Therefore, the one-point turbulence modeling is a promising approach to the abstraction of primary characteristics of large-scale motions in a turbulent flow. The necessity of proper turbulence models in the study of astrophysical or geophysical phenomena is clear because of their huge spatial scales [1]. A shortcoming of one-point modeling is that small-scale components closely connected with rapid time variation are generally beyond its scope.

The current turbulence models are classified roughly into two categories. One is the models based on approximate expressions for the Reynolds stress using the eddy-viscosity concept or its extended form. Another is the second-order models, in which the pressure- velocity-strain correlation in the Reynolds-stress transport equation plays a central role. These two types of models have been constructed by making full use of the invariance principles intrinsic to the Navier-Stokes equation as well as dimensional and tensor analyses [2,3].

Over the past ten years, some progress has been made in the study of turbulence models based on the application of two-point closure methods, such as the two-scale direct-interaction approximation (TSDIA) [4–6], the renormalization-group (RNG) method [7–9], etc. A typical property of these methods is that they give an asymptotic expansion for the Reynolds stress with the isotropic-eddy-viscosity representation as the leading term. The similar situation is also encountered in the analysis of the pressure- velocity-strain correlation in the Reynolds-stress transport equation [10]. As a result, some additional theoretical devices are necessary for deriving the counterpart in the second-order modeling (for instance, see [11,12]). As an effect that has been missing in the current turbulence modeling, the importance of helicity effects on the Reynolds stress has been pointed out recently by the author and Yokoi [13,14] in close relation to effects of vorticity. Specifically, the turbulent helicity was shown to control the energy cascade processes and be an important measure of duration of large-scale three-dimensional flow structures.

The above-stated developments in the turbulence modeling have been made rather independently in the studies of the eddy-viscosity-type modeling, the second-order modeling, and the two-point closure methods. In this work, we shall bridge a gap between the first two methods through the intermediation of the last, specifically, using the results of the TSDIA. As a result, we shall show that the findings in the eddy-viscosity-type modeling can be used in the second-order modeling. The present paper is organized as follows. The fundamental equations are given in Sec. II. In Sec. III, a Reynolds-stress transport equation with the effect of helicity included is derived through the renormalization of a generalized eddy-viscosity-type expression for the Reynolds stress that was obtained using the TSDIA. In Sec. IV, a higher-order eddy-viscosity-type expression for the Reynolds stress with the frame-rotation effect incorporated is derived through the application of an iterative approximation to the Reynolds-stress transport equation. The role of each term in the resulting expression is discussed to clarify some prominent features of turbulent flows. In Sec. V, a rotating channel flow is investigated as an example for confirming the usefulness of the present method. In Sec. VI, the familiar model for the isotropic eddy viscosity is extended to the representation that includes effects of trajectory appearing in flows under an adverse pressure gradient, along a curved boundary, etc. The concluding remarks are given in Sec. VII.

II. FUNDAMENTAL EQUATIONS

Throughout this work, we consider a fluid motion at very low Mach numbers to neglect fluid compressibility. Such a motion of a viscous fluid is described by the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\nu\Delta\mathbf{u} , \tag{1}$$

with the solenoidal condition $\nabla\cdot\mathbf{u}=0$, where $\mathbf{u}$ is the velocity, p is the pressure divided by fluid density, and ν is the kinematic viscosity.

In the turbulence modeling of one-point type, attention is focused on large-scale components of a turbulent motion. To this end, we use the ensemble mean $\langle\ \rangle$ to divide a quantity f into the mean $\langle f\rangle$ and the fluctuation f' as

$$f=F+f',\quad F=\langle f\rangle , \tag{2}$$

where $f=(\mathbf{u},p,\boldsymbol{\omega})$, $F=(\mathbf{U},P,\boldsymbol{\Omega})$, and $f'=(\mathbf{u}',p',\boldsymbol{\omega}')$ [$\boldsymbol{\omega}$ $(=\nabla\times\mathbf{u})$ is the vorticity]. The mean velocity $\mathbf{U}$ obeys the equation

$$\frac{D\mathbf{U}}{Dt}=-\nabla P+\nabla\cdot R+\nu\Delta\mathbf{U} , \tag{3}$$

where $D/Dt=\partial/\partial t+(\mathbf{U}\cdot\nabla)$ and R is the Reynolds stress defined by

$$R_{ij}=-\langle u_i'u_j'\rangle \tag{4}$$

[$(\nabla\cdot R)_i=(\partial/\partial x_j)R_{ji}$]. On the other hand, the equation for $\mathbf{u}'$ is

$$\frac{D\mathbf{u}'}{Dt}+(\mathbf{u}'\cdot\nabla)\mathbf{U}+\nabla\cdot(\mathbf{u}'\mathbf{u}'+R)=-\nabla p'+\nu\Delta\mathbf{u}' , \tag{5}$$

with $\nabla\cdot\mathbf{u}'=0$.

For the later discussion of the second-order modeling, we give the transport equation for the Reynolds stress. From Eq. (5), we have

$$\frac{DR_{ij}}{Dt}=-R_{ik}\left[\frac{\partial U_j}{\partial x_k}\right]-R_{jk}\left[\frac{\partial U_i}{\partial x_k}\right]-\Pi_{ij}+\epsilon_{ij}-\left[\frac{\partial}{\partial X_k}\right]T_{ijk}+\nu\Delta R_{ij} , \tag{6}$$

where Π_{ij}, etc., are defined as

$$\Pi_{ij}=\left\langle p'\left[\frac{\partial u_j'}{\partial x_i}+\frac{\partial u_i'}{\partial x_j}\right]\right\rangle , \tag{7}$$

$$\epsilon_{ij}=2\nu\left\langle\left[\frac{\partial u_i'}{\partial x_k}\right]\left[\frac{\partial u_j'}{\partial x_k}\right]\right\rangle , \tag{8}$$

$$T_{ijk}=-[\langle u_i'u_j'u_k'\rangle+\langle p'u_i'\rangle\delta_{jk}+\langle p'u_j'\rangle\delta_{ik}] . \tag{9}$$

Specifically, the turbulent energy K obeys

$$\frac{DK}{Dt}=P_k-\epsilon+\nabla\cdot\mathbf{T}_K+\nu\Delta K , \tag{10}$$

where

$$K=\langle\mathbf{u}'^2/2\rangle , \tag{11}$$

$$P_K=-\langle u_i'u_j'\rangle\frac{\partial U_j}{\partial x_i} , \tag{12}$$

$$\epsilon=\nu\left\langle\left[\frac{\partial u_j'}{\partial x_i}\right]^2\right\rangle , \tag{13}$$

$$\mathbf{T}_K=-\left\langle\left[\frac{\mathbf{u}'^2}{2}+p'\right]\mathbf{u}'\right\rangle . \tag{14}$$

Here we should make the following points. One is the fact that the pressure- velocity-strain correlation Π_{ij} does not contribute to the K equation (10), but it plays a key role in the partition of energy among three components $\langle u_1'^2\rangle$, $\langle u_2'^2\rangle$, and $\langle u_3'^2\rangle$. Namely, Π_{ij} is closely associated with the degree of anisotropy in a turbulent motion. Many of the shortcomings of eddy-viscosity-type models are considered to arise from the fact that these models are not directly linked with Π_{ij}. Another point to be made is linked with the effect of $(D/Dt)R_{ij}$ in Eq. (6). The Lagrange derivative D/Dt describes convection or trajectory effects (in this paper, we shall use the latter terminology). As a result, $(D/Dt)R_{ij}$ is important in expressing these effects on turbulent intensities. Their typical examples are effects of boundary curvature and adverse pressure gradient. This point is also a primary cause of the shortcomings of eddy-viscosity-type models.

III. DERIVATION OF REYNOLDS-STRESS TRANSPORT EQUATION

A goal of the second-order modeling is to relate Π_{ij} [Eq. (7)] and T_{ijk} [Eq. (9)], etc. to R_{ij}, $\mathbf{U}$, etc. and close Eq. (6). In Eq. (6), Π_{ij} is specifically important, since it has great influence on the anisotropy of turbulent intensities, as has already been noted. In its modeling, much attention has really been paid to the satisfaction of the constraints and invariance properties intrinsic to the Navier-Stokes equations. Such representative models are the models of Launder, Reece, and Rodi [15], Speziale, Sarkar, and Gatski [16], Shih, Chen, and Lumley [17], etc. Two-point closure methods have also been applied to the investigation of Π_{ij}, as in the works of Weinstock [18,19] and Rubinstein and Barton [11].

On the other hand, the author [12] has recently proposed a new method of deriving a model Reynolds-stress transport equation through the investigation of an eddy-viscosity-type expression for the Reynolds stress itself using the TSDIA. A merit of this method lies in the fact that the two-point analysis is made of the second-order correlation R_{ij}, but not of the third-order ones like Π_{ij}. In what follows, we shall summarize the above method and at the same time extend it to incorporate the effect of turbulent helicity into a model Reynolds-stress transport equation.

A. TSDIA results for the Reynolds stress

We first combine the previous TSDIA higher-order expression [4] for the Reynolds stress with the recent results [13,14] concerning the effect of turbulent helicity on the stress. As a result, the deviatoric part of the Reynolds stress B_{ij}, which is defined by

$$B_{ij}=R_{ij}+\tfrac{2}{3}K\delta_{ij} \ , \tag{15}$$

is written as

$$B_{ij}=\sum_{n=1}(B_{ij})_n \ . \tag{16}$$

Here the first few terms are given by

$$(B_{ij})_1\equiv(B_{ij})_{E,S}=\nu_{E,S}S_{ij} \ , \tag{17}$$

$$(B_{ij})_2\equiv(B_{ij})_{E,U}=C_{E,U}(K/\epsilon)\left[\left[\frac{D}{Dt}\right]\nu_{E,S}\right]S_{ij}+R_{E,U} \ , \tag{18}$$

$$(B_{ij})_3\equiv(B_{ij})_H=-C_H(1/\epsilon)\nu_{E,S}^2\left[\Omega_i\left[\frac{\partial H}{\partial x_j}\right]+\Omega_j\left[\frac{\partial H}{\partial x_i}\right]-\tfrac{2}{3}\boldsymbol{\Omega}\cdot\nabla H\delta_{ij}\right] , \tag{19}$$

$$(B_{ij})_4\equiv(B_{ij})_N=-C_{N1}(K/\epsilon)\nu_{E,S}(S_{ik}S_{kj}+S_{jk}S_{ki}-\tfrac{2}{3}S{:}S\delta_{ij})-C_{N2}(K/\epsilon)\nu_{E,S}(S_{ik}\Omega_{kj}+S_{jk}\Omega_{ki}) \\ +C_{N3}(K/\epsilon)\nu_{E,S}(\Omega_{ik}\Omega_{kj}-\tfrac{1}{3}\Omega{:}\Omega\delta_{ij}) \ , \tag{20}$$

$$(B_{ij})_5\equiv(B_{ij})_c=-C_c(K/\epsilon)\left[\frac{D}{Dt}\right](\nu_{E,S}S_{ij}) \ , \tag{21}$$

$$(B_{ij})_6\equiv(B_{ij})_T=C_c(K/\epsilon)\nabla\cdot T_{Bij} \ , \tag{22}$$

with the definition $A{:}B=A_{ij}B_{ji}$. The eddy viscosity $\nu_{E,S}$, the mean velocity-strain tensor S_{ij}, the mean vorticity tensor Ω_{ij}, and the turbulent helicity H are defined as

$$\nu_{E,S}=C_{E,S}K^2/\epsilon \ , \tag{23}$$

$$S_{ij}=\frac{\partial U_j}{\partial x_i}+\frac{\partial U_i}{\partial x_j} \ , \tag{24}$$

$$\Omega_{ij}=\frac{\partial U_j}{\partial x_i}-\frac{\partial U_i}{\partial x_j} \ , \tag{25}$$

$$H=\langle u'\cdot\omega'\rangle \ , \tag{26}$$

respectively. Moreover, $R_{E,U}$ in Eq. (18) is given by

$$R_{E,U}=\left[C'_{E,U}(K^3/\epsilon^3)\frac{D\epsilon}{Dt}\right]S_{ij} \ , \tag{27}$$

and T_{Bij} represents the transport effect, which will be referred to later.

In the above expressions, numerical factors $C_{E,U}$, C_{N1}, C_{N2}, C_{N3}, C_C, $C_{E,S}$ and $C'_{E,U}$ in Eqs. (17), (18), (19)–(23), and (27) are estimated as

$$C_{E,U}=0.320, C_{N1}=0.0765, \quad C_{N2}=0.015 \ , \\ C_{N3}=0.183, \quad C_C=0.399 \ , \\ C_{E,S}\equiv0.0785, \quad C'_{E,U}=0.026 \ , \tag{28}$$

within the framework of the TSDIA. These values deviate from the ones optimized through the application to some fundamental turbulent shear flows, but they can be considered to give a measure of their magnitude. On the other hand, C_H in Eq. (19) is optimized as

$$C_H=0.37 \tag{29}$$

from the application to a swirling pipe flow [14,20]. The capital subscripts E,S, E,U, H, N, C, and T in $(B_{ij})_n$ denote eddy viscosity (the steady part), eddy viscosity (the unsteady part), helicity, the nonlinear part, convection, and transport, respectively.

Before proceeding to the derivation of a Reynolds-stress transport equation, we give a brief account of the mathematical structure of the present asymptotic expansion for the Reynolds stress. A primary feature of the TSDIA results is that they are obtained in an order of derivatives of **U**, K, etc. Namely, the first-order analysis of B_{ij} gives Eq. (17) or the familiar isotropic-eddy-viscosity representation that depends linearly on the first-order derivatives of **U**. The second-order analysis leads to Eqs. (18)–(21), which consist of the terms dependent on the second-order derivatives or the terms quadratic in the first-order derivatives. Equation (22), in which T_{Bij} includes ΔS_{ij}, comes from the third-order analysis. The derivation of Eq. (20) was also done by Rubinstein and Barton [8] using the RNG method.

Some of $(B_{ij})_n$ ($n=1$–6) have already been applied to the analysis of turbulent flows and their roles have been clarified. For example, the nonlinear term $(B_{ij})_N$ [Eq. (20)] can produce secondary flows in a square-duct flow, which arise from the anisotropy of turbulent intensities in the cross section of the square duct and are beyond the capability of the isotropic eddy-viscosity representation (17) [21,22]. In this context, Speziale [3] recommended to drop the Ω-Ω terms in Eq. (20) from consistency with rotating isotropic turbulence. The retention of these terms results in the evolution of an initially isotropic turbulent state to an anisotropic one under the effect of a rigid-body rotation. This point contradicts the findings of the full numerical simulation of Bardina, Ferziger, and Rogallo [23].

B. Renormalization of an eddy-viscosity-type representation

Equation (16) with Eqs. (17)–(22) is an asymptotic expansion for B_{ij} that was constructed in the combination of the Navier-Stokes equation (1) and the TSDIA. Such an expansion may be considered a solution of a model equation for B_{ij}. Therefore, it is anticipated that a Reynolds-stress transport equation can be obtained by seeking the equation satisfying Eq. (16) with Eqs. (17)–(22). From such a model equation, we can see how Π_{ij}, etc. in the Reynolds-stress transport equation (6) are modeled.

A method of deriving a governing equation leading to the asymptotic solution (16) is what is called a renormalization procedure [12]. In Eq. (16), we have only the first few terms in the infinite asymptotic series. In order to infer the whole series from this restricted information, we perform the partial but infinite summation of the terms in the asymptotic expansion (16). The simplest method for such a summation is to replace $\nu_{E,S}S_{ij}$ in B_{ij}, except $(B_{ij})_1$, with B_{ij} (note that $\nu_{E,S}S_{ij}$ is the leading term in the asymptotic expansion for B_{ij}). As a result, we obtain

$$B_{ij}=\chi_{ij}-C_C(K/\epsilon)\left[\left[\frac{D}{Dt}\right]B_{ij}-\nabla\cdot T_{Bij}\right], \tag{30}$$

where

$$\chi_{ij}=\nu_{E,S}S_{ij}+C_{E,U}(K/\epsilon)\left[\left[\frac{D}{Dt}\right]\nu_{E,S}\right]S_{ij}+C_H(1/\epsilon)\nu_{E,S}^2\left[\Omega_i\left[\frac{\partial H}{\partial x_j}\right]+\Omega_j\left[\frac{\partial H}{\partial x_i}\right]-\tfrac{2}{3}\mathbf{\Omega}\cdot\nabla H\delta_{ij}\right]$$
$$-C_{N1}(K/\epsilon)(B_{ik}S_{kj}+B_{jk}S_{ki}-\tfrac{2}{3}B:S\delta_{ij})-C_{N2}(K/\epsilon)(B_{ik}\Omega_{kj}+B_{jk}\Omega_{ki})+C_{N3}(K/\epsilon)\nu_{E,S}(\Omega_{ik}\Omega_{kj}-\tfrac{1}{3}\Omega:\Omega\delta_{ij}) . \tag{31}$$

Here we have neglected $R_{E,U}$ [Eq. (27)] in $(B_{ij})_2$ [Eq. (18)] since the primary effect of $D\epsilon/Dt$ has already been incorporated through $(D/Dt)\nu_{E,S}$.

Equation (30) may be rewritten as

$$\left[\frac{D}{Dt}\right]B_{ij}=-\lambda B_{ij}+\lambda\chi_{ij}+\nabla\cdot T_{Bij} , \tag{32}$$

where

$$\lambda=(1/C_C)(\epsilon/K) . \tag{33}$$

Namely, we have reached a model transport equation for B_{ij} (the deviatoric part of R_{ij}). The comparison between Eq. (32) and the exact equation (6) shows that the present model equation (32) with Eq. (31) is equivalent to

$$\Pi_{ij}=(\Pi_{ij})_A+(\Pi_{ij})_B , \tag{34}$$

with $\epsilon_{ij}=2\epsilon\delta_{ij}$, where

$$(\Pi_{ij})_A=C_S(\epsilon/K)B_{ij}+C_{R0}KS_{ij}-C_{R1}(B_{ik}S_{kj}+B_{jk}S_{ki}-\tfrac{2}{3}B:S\delta_{ij})-C_{R2}(B_{ik}\Omega_{kj}+B_{jk}\Omega_{ki}) , \tag{35}$$

$$(\Pi_{ij})_B=-C_H'(1/K)\nu_{E,S}^2\left[\Omega_i\left[\frac{\partial H}{\partial x_j}\right]+\Omega_j\left[\frac{\partial H}{\partial x_i}\right]-\tfrac{2}{3}\mathbf{\Omega}\cdot\nabla H\delta_{ij}\right]-C_{E,U}''\left[\left[\frac{D}{Dt}\right]\nu_{E,S}\right]S_{ij} . \tag{36}$$

Here we should note that the Ω-Ω terms in Eq. (31) were dropped for the sake of consistency with isotropic rotating turbulence. Model constants C_S, etc., are related to $C_{E,U}$, etc. in Eqs. (18)–(23) as

$$C_S=1/C_C,\quad C_{R0}=\tfrac{2}{3}-C_S/C_C,\quad C_{R1}=\tfrac{1}{2}-C_{N1}/C_C ,$$
$$C_{R2}=\tfrac{1}{2}-C_{N2}/C_C,\quad C_H'=C_H/C_C,\quad C_{E,U}''=C_{E,U}/C_C . \tag{37}$$

Let us refer to an equation for the turbulent helicity H, which is necessary for closing both the present eddy-viscosity-type and second-order models. Such a model equation has already been discussed in Ref. [14] and given as

$$\frac{DH}{Dt}=R_{ij}\left[\frac{\partial\Omega_j}{\partial x_i}\right]-\Omega_i\left[\frac{\partial}{\partial x_i}\right]R_{ji}-C_H(\epsilon/K)H+\nabla\cdot[K\mathbf{\Omega}+(\nu_{E,S}/\sigma_H)\nabla H] . \tag{38}$$

Here the terms except the C_H- and $(\nu_{E,S}/\sigma_H)$-related ones can be derived exactly using Eq. (5). The model constants C_H and σ_H have been optimized as 1.5 and 1.6, respectively, through the study of a decaying swirling pipe flow [14,20].

C. Relationship with the conventional models

In Eq. (34), the first part $(\Pi_{ij})_A$ corresponds to the model of Launder, Reece, and Rodi [15], which is the prototype of various models of Π_{ij} in the second-order modeling. In the modeling, the first B_{ij}-related term and the rest are often called the slow and rapid terms, respectively. The simplest of the model of Launder, Reece, and Rodi is given by $C_{R1}=C_{R2}$ (=0.3). The model cannot satisfy the equilibrium state of homogeneous shear turbulence and the realizability condition guaranteeing the positive definiteness of turbulent intensities. In order to improve these deficiencies, Eq. (35) is extended to include the terms up to the second order in B_{ij} with the coefficients dependent on $B{:}B$ and $B_{ij}B_{jk}B_{ki}$ [16,17]. Such modeling of Π_{ij} corresponds to the renormalization of the asymptotic expansion for B_{ij} [Eq. (16)] with higher-order terms included.

In the second part $(\Pi_{ij})_B$ with Eq. (36), the helicity-related terms correspond to $(B_{ij})_H$ [Eq. (19)]. Within the framework of the eddy-viscosity-type approximation, $(B_{ij})_H$ plays a central role in the study of the effect of swirling or large-scale vortices in a pipe flow [14,20]. Such an effect cannot be properly dealt with using $(B_{ij})_N$ [Eq. (20)], which leads to the B-S and B-Ω terms in $(\Pi_{ij})_A$ and includes the mean-vorticity effect, since

$$\Omega_{ij}=\epsilon_{ijk}\Omega_k, \quad \Omega_i=\tfrac{1}{2}\epsilon_{ijk}\Omega_{jk} \ . \tag{39}$$

Namely, $(B_{ij})_H$ cannot be substituted by $(B_{ij})_N$. This fact signifies that the helicity-related terms in $(\Pi_{ij})_B$, which arise from $(B_{ij})_H$, cannot be substituted by the B-S and B-Ω terms in $(\Pi_{ij})_A$ corresponding to $(B_{ij})_N$. Therefore, we may conclude that effects of helicity should also be considered in the second-order modeling.

The $[(D/Dt)\nu_{E,S}]$-related term in $(\Pi_{ij})_B$ [Eq. (36)] represents an effect of flow trajectory on Π_{ij}. Using Eqs. (10), (23), and the model equation for ϵ,

$$\frac{D\epsilon}{Dt}=C_{D1}(\epsilon/K)P_K-C_{D2}\epsilon^2/K+\nabla\cdot T_D \ , \tag{40}$$

we have

$$-C''_{E,U}\left[\left[\frac{D}{Dt}\right]\nu_{E,S}\right]S_{ij}=[C'_{R0}+C''_{R0}(P/\epsilon)]KS_{ij} \ , \tag{41}$$

with $C'_{R0}=C''_{E,U}C_{E,S}(2-C_{D2})$ and $C''_{R0}=-C''_{E,U}C_{E,S}(2-C_{D1})$. Here C_{D1} ($\simeq 1.4$) and C_{D2} ($\simeq 1.9$) are model constants, and the ϵ transport rate $\mathbf{T}_D$ as well as the K counterpart $\mathbf{T}_K$ [Eq. (14)] have been dropped for simplicity of discussion. Therefore, when the $[(D/Dt)\nu_{E,S}]$-related term is retained, C_{R0} in $(\Pi_{ij})_A$ [Eq. (35)] effectively changes into $C_{R0}+C'_{R0}+C''_{R0}(P_K/\epsilon)$ [24], where $P_K=B_{mn}S_{nm}/2$. In the limit of vanishing B_{ij}, the C''_{R0}-related part disappears and does not violate the lowest-order rapid-distortion constraint where $\Pi_{ij}=\frac{4}{5}KS_{ij}$.

Finally, let us refer to the transport term T_{Bij} in $(B_{ij})_T$ [Eq. (22)]. From the TSDIA [3], $(B_{ij})_T$ is given by

$$(B_{ij})_T=C(K/\epsilon)\nu_{E,S}^2\Delta S_{ij} \ , \tag{42}$$

which is rewritten as

$$(B_{ij})_T=C_C(K/\epsilon)\nabla\cdot[(\nu_{E,S}/\sigma_T)\nabla(\nu_{E,S}S_{ij})]+R_T \ , \tag{43}$$

where the residual terms R_T depend on $\nabla\nu_{E,S}$ and ∇S_{ij}. In Eqs. (42) and (43), C and σ_T are numerical constants and are related to C_C in Eq. (21) as $C_C=C/\sigma_T$. The simplest renormalization of Eq. (43) gives

$$(B_{ij})_T=C_C(K/\epsilon)\nabla\cdot[(\nu_{E,S}/\sigma_T)\nabla B_{ij}] \tag{44a}$$

or

$$T_{Bij}=(\nu_{E,S}/\sigma_T)\nabla B_{ij} \tag{44b}$$

[see Eq. (22)]. Equation (44) should be compared with the model [15]

$$T_{Bijk}=-C_T(K/\epsilon)R_{km}\left[\frac{\partial}{\partial x_m}\right]B_{ij} \ , \tag{45a}$$

which reduces to

$$T_{Bij}=\tfrac{2}{3}(C_T/C_{E,S})\nu_{E,S}\nabla B_{ij} \ , \tag{45b}$$

under the isotropic-diffusivity approximation that $R_{km}\simeq-\frac{2}{3}K\delta_{km}$ (C_T is a constant). Therefore, Eq. (44) obtained using the TSDIA is the simplest model of the transport effect on B_{ij}.

IV. DERIVATION OF A HIGHER-ORDER EDDY-VISCOSITY-TYPE EXPRESSION

In the previous section, we bridged a gap between the eddy-viscosity-type and second-order models through the application of the renormalization procedure to the TSDIA results. From the standpoint of the TSDIA, it is very difficult to derive terms that are of higher order than $(B_{ij})_n$ ($n=1$–6) because of the mathematical complexity. Such higher-order effects, however, can become important in analyzing various types of turbulent flows on the basis of the eddy-viscosity-type approximation to the Reynolds stress. Such a typical example is a turbulent flow between a rotating channel.

A method for overcoming the above-stated difficulty with the TSDIA is to use the second-order model (30) that has been constructed using the TSDIA and the renormalization procedure. Namely, we solve Eq. (30) iteratively with $B_{ij}=\nu_{E,S}S_{ij}$ as the lowest-order solution. In the first iteration, B_{ij} on the right-hand side of Eq. (30) is simply replaced with $\nu_{E,S}S_{ij}$. At this time, we move to the frame that is rotating with the angular velocity $\boldsymbol{\omega}_F$. In such a frame, the mean-vorticity tensor Ω_{ij} and the mean vorticity $\boldsymbol{\Omega}$ are replaced with the intrinsic counterparts [3]

$$\Omega_{Iij}=\Omega_{ij}+2\epsilon_{ijk}\omega_{Fk} \ , \tag{46a}$$

$$\boldsymbol{\Omega}_I=\boldsymbol{\Omega}+2\boldsymbol{\omega}_F \ , \tag{46b}$$

respectively. As a result, we obtain

$$B_{ij}=(B_{ij})_{E,S}+(B_{ij})_{E,U}+(B_{ij})_H+(B_{ij})_N+R_B \ , \tag{47}$$

where $(B_{ij})_{E,S}$ and $(B_{ij})_{E,U}$ are given by Eqs. (17) and (18), respectively [$R_{E,U}$ in Eq. (18) was dropped], and

$$(B_{ij})_H=-C_H(1/\epsilon)\nu_{E,S}^2\left[\Omega_{Ii}\left[\frac{\partial H}{\partial x_j}\right]+\Omega_{Ij}\left[\frac{\partial H}{\partial x_i}\right]-\tfrac{2}{3}\boldsymbol{\Omega}_I\cdot\nabla H\delta_{ij}\right] \ , \tag{48}$$

$$(B_{ij})_N = -C_{N1}(K/\epsilon)\nu_{E,S}(S_{ik}S_{kj}+S_{jk}S_{ki}-\tfrac{2}{3}S{:}S\delta_{ij}) - C_{N2}(K/\epsilon)\nu_{E,S}(S_{ik}\Omega_{Ikj}+S_{jk}\Omega_{Iki}) \ . \qquad (49)$$

The residual terms R_B consist of the convection effect of $(B_{ij})_c$ and the transport effect $(B_{ij})_T$.

In $(B_{ij})_N$ [Eq. (49)], the frame rotation ($\boldsymbol{\omega}_F$) effect is included in the $\boldsymbol{\Omega}_I$-related terms. This effect has already been examined using the TSDIA [25], but it is not sufficient for describing important properties associated with the anisotropy of turbulent intensities that is generated by effects of frame rotation. This point will be detailed in the later discussion of a rotating channel flow. In order to overcome this difficulty, we proceed to the second iteration. In the iteration, we have many new terms, but our attention is focused on the $\boldsymbol{\Omega}_I$ effect closely connected with frame rotation. As a result, we have

$$B_{ij} = (B_{ij})_E + (B_{ij})_H + (B_{ij})_N + (B_{ij})_{N,\mathrm{VF}} + R_B \ , \qquad (50)$$

where

$$(B_{ij})_E = \nu_{E,S}\left[1 + C_{TE}(K/\epsilon)\left[\frac{D}{Dt}\right]\ln\nu_{E,S}\right]^{-1} S_{ij} \ , \qquad (51)$$

$$(B_{ij})_{N,\mathrm{VF}} = -C_{N2}(K/\epsilon)[(B_{ik})_N\Omega_{Ikj} + (B_{jk})_N\Omega_{Iki}] \ , \qquad (52)$$

and R_B consist of part of $(B_{ij})_c$, $(B_{ij})_T$, and other higher-order terms. The constant C_{TE} given by

$$C_{TE} = C_C - C_{E,U} \qquad (53)$$

is positive from Eq. (28). The eddy-viscosity term $(B_{ij})_E$ will be discussed below.

Equation (52) for $(B_{ij})_{N,\mathrm{VF}}$ arises from the replacement of B_{ij} in the C_{N2}-related terms of Eq. (31) with $(B_{ij})_N$. On the other hand, the Ω_I-related terms in $(B_{ij})_N$ [Eq. (49)] come from the replacement of B_{ij} in the C_{N2}-related terms of Eq. (31) with the isotropic-eddy-viscosity representation $(B_{ij})_{E,S}$ [Eq. (17)]. Since $(B_{ij})_N$ is closely associated with anisotropy of turbulent intensities, $(B_{ij})_{N,\mathrm{VF}}$ represents the interaction between anisotropy of turbulent intensities and effects of vorticity and frame rotation (VF). It is difficult to calculate higher-order terms like $(B_{ij})_{N,\mathrm{VF}}$ using the TSDIA, but they can be easily obtained using the above method. This point is a merit of such a method.

Equation (51) is a combination of the steady-eddy-viscosity approximation $(B_{ij})_{E,S}$ [Eq. (17)], the unsteady counterpart $(B_{ij})_{E,U}$ [Eq. (18)], and $-C_C(D\nu_{E,S}/Dt)S_{ij}$ that is part of $(B_{ij})_C$ [Eq. (21)]. Within the framework of the TSDIA, $(B_{ij})_{E,S}$ is the lowest- or first-order term in the asymptotic expansion for B_{ij}, whereas $(B_{ij})_{E,U}$ and $(B_{ij})_C$ are the second-order one implying the correction terms to the former. In this original derivation, these corrections are valid only when they are much smaller than $(B_{ij})_{E,S}$. When these two effects are incorporated into the current turbulence models like the K-ϵ model, their excessive growth in a narrow region often leads to numerical instability and gives rise to wrong effects on the entire flow region. A method for reducing such effects is to extend the present expression to a more general functional form whose asymptotic expansion includes $(B_{ij})_{E,S}$, $(B_{ij})_{E,U}$, and part of $(B_{ij})_C$ as the leading three terms. Equation (48) is the simplest candidate. In reality, Shimomura [26] incorporated effects of magnetic fields on a subgrid-scale model in a similar form to succeed in explaining the turbulence suppression mechanism by magnetic fields. The relationship of $(B_{ij})_E$ with effects of an adverse pressure gradient will be referred to later.

V. ROTATING CHANNEL FLOW

Equation (50) gives some important corrections to the isotropic eddy-viscosity representation $(B_{ij})_E$ [Eq. (51)]. The second term $(B_{ij})_H$ contains the effect of turbulent helicity connected with the mean vorticity and is appropriate for describing helical flow structures. Its importance was really confirmed in the study of a swirling pipe flow [14,20]. The third term $(B_{ij})_N$ expresses nonlinear effects of the mean velocity gradient, which is linked with the anisotropy of turbulent intensities in a shear flow. This point was also confirmed in the studies of channel flow, secondary flows in a square-duct flow, etc. [21,22,27].

In various engineering and natural sciences, effects of frame rotation often become very important. They are usually connected with other effects and are generally difficult to abstract in a simple form. One of the few examples showing effects of frame rotation in a clear form is a turbulent flow in a rotating channel. Here we consider that two walls are parallel to the x axis and are located at $y=\pm D_{\mathrm{CH}}/2$ (D_{CH} is the width of the channel). The channel is rotating with the angular velocity vector $\boldsymbol{\omega}_F$ that is along the z direction. In this flow situation, the mean velocity and the angular velocity are written as $(U(y),0,0)$ and $(0,0,\omega_{FZ})$, respectively. The rotating channel flow is important not only for the above-stated reason but also from the standpoint of mechanical engineering. Many mechanical-engineering flows related to turbine blades really have rotating-channel-like properties.

A typical property of a rotating channel flow is asymmetry of the mean velocity, the shear stress, the turbulent energy, etc. with respect to the center line. These asymmetries are linked with one another. For example, the turbulent energy K is intensified near the lower wall, whereas it is weakened near the upper wall. This property can be understood intuitively considering the Coriolis force. Under the force, fluid is pushed towards the lower wall, near which the mean velocity gradient steepens. On the other hand, the Coriolis force tends to pull fluid from the upper wall and the mean velocity gradient becomes more gentle. These effects lead to the above-stated property of K.

In the foregoing flow geometry, $(B_{ij})_H$ [Eq. (48)] and $(B_{ij})_N$ [Eq. (49)] do not contribute to R_{xy} $(=-\langle u'v'\rangle)$ (u' and v' are the velocity fluctuations in the x and y directions, respectively). As a result, the direct effect of $\boldsymbol{\omega}_F$ on $\langle u'v'\rangle$ cannot be explained using $(B_{ij})_H$ and $(B_{ij})_N$, although the $\boldsymbol{\omega}_F$-related terms are included in them. At the present stage, any eddy-viscosity-type approximation has not succeeded in the proper treatment of a rotating channel flow (the attempt of Shimoura [28] will be referred to later). The second-order model [29] and the large eddy simulation [30] based on the subgrid-scale modeling can overcome this difficulty. Specifically, the cause of the success in the former lies in the Coriolis effect on the Reynolds-stress transport equation. It is written as

$$C_{ij}=2(\epsilon_{ikm}\omega_{Fm}R_{kj}+\epsilon_{jkm}\omega_{Fm}R_{ki}) \, , \tag{54}$$

which is added to the right-hand side of Eq. (6). When we approximate R_{ij} in Eq. (54) by the isotropic-eddy-viscosity representation $(B_{ij})_E$, the resulting expression is very similar to the $\boldsymbol{\omega}_F$-related part in $(B_{ij})_N$. In the present flow situation, such a part does not contribute to R_{xy}, as was noted above. This fact signifies that the effect of frame rotation in a rotating channel flow is closely associated with the interaction between the frame-rotation effect and the anisotropy of turbulent intensities, as will become clearer.

The $\boldsymbol{\omega}$-related part of $(B_{ij})_{N,\mathrm{VF}}$ [Eq. (52)], which is denoted by $(B_{ij})_{N,F}$, is given by

$$(B_{ij})_{N,F}=2C_{N2}(K/\epsilon)[\epsilon_{ikm}\omega_{Fm}(B_{kj})_N+\epsilon_{jkm}\omega_{Fm}(B_{ki})_N] \, . \tag{55}$$

The contribution of Eq. (55) to R_{xy} $(=B_{xy})$ is

$$2C_{N2}(K/\epsilon)\omega_{FZ}[(B_{yy})_N-(B_{xx})_N] \, . \tag{56}$$

We use Eq. (49) to obtain

$$(B_{xx})_N=-(K/\epsilon)\nu_{E,S}\left[\frac{dU}{dy}\right]^2(\tfrac{2}{3}C_{N1}+2C_{N2}) \, , \tag{57a}$$

$$(B_{yy})_N=-(K/\epsilon)\nu_{E,S}\left[\frac{dU}{dy}\right]^2(\tfrac{2}{3}C_{N1}-2C_{N2}) \, . \tag{57b}$$

Therefore R_{xy} is given by

$$R_{xy}=\nu_{E,S}\left[\frac{dU}{dy}\right]+8C_{N2}^2(K/\epsilon)^2\nu_{E,S}\omega_{FZ}\left[\frac{dU}{dy}\right]^2 . \tag{58}$$

In Eq. (58), the first term is positive and negative near the lower and upper walls, respectively, whereas the second term is always positive. As a result, R_{xy} is asymmetric with respect to the center line ($y=0$). The production rate of K, which is defined by $P_K=R_{xy}(dU/dy)$, is written as

$$P_K=\nu_{E,S}\left[\frac{dU}{dy}\right]^2+8C_{N2}^2(K/\epsilon)^2\nu_{E,S}\omega_{FZ}\left[\frac{dU}{dy}\right]^3 . \tag{59}$$

Equation (59) clearly shows that the $\boldsymbol{\omega}_F$ effect arising from the second term enhances the turbulent-energy production near the lower wall, whereas the ω_F effects weakens it near the upper wall. This result is consistent with the foregoing observation of K.

Nisizima [31] incorporated the new term $(B_{ij})_{N,F}$ into the turbulence model [22] of K-ϵ type to examine a rotating channel flow with the Reynolds number R $(=U_B D_{\mathrm{CH}}/\nu)\simeq 35\,000$ and the rotation number R_0 $(=\omega_{FZ}D_{\mathrm{CH}}/U_B)=(0,0.068,0.2)$ (U_B is the bulk mean velocity). Nisizima found that $(B_{ij})_{N,F}$ can really produce the asymmetry of the mean velocity and the turbulent energy with respect to the center line. At the same time, it was found that the addition of

$$C_{DF}K\epsilon_{ijk}\Omega_{Ii}(\partial U_k/\partial x_j) \tag{60}$$

to the right-hand side of the ϵ equation (40) leads to better reults, where C_{DF} is a model constant (see Ref. [31] for the details of numerical computation). This additional effect was originally pointed out by Shimomura [28], who used a method [4] deriving a model ϵ equation with the aid of the TSDIA results.

As an example illustrating the performance of the present model, the comparison between the computational [31] and experimental [32] results about the mean velocity and the shear stress is given in Figs. 1 and 2. These computational results show that the important features of a rotating channel flow can be captured within the framework of the eddy-viscosity-type approximation with the $\boldsymbol{\omega}_F$ effect incorporated, although there is some room for improvement of the quantitative accuracy of the results.

Here we should note the additional term given by Eq. (60). The inclusion of this term in the ϵ equation (40) leads to better results, at least in the analysis of a rotating channel flow. The new term, however, eliminates $\epsilon/K=0$ as a fixed point in the dynamical system, constituted by the present system of K-ϵ type [3]. As a result, the simple addition of Eq. (60) leads to a lack of the ability to predict restabilization in the case of homogeneous turbulent shear flow in a rotating frame, which was examined in detail by Speziale and Mhuiris [3,33]. In order to remove this shortcoming and retain the $\boldsymbol{\omega}_F$ effect on

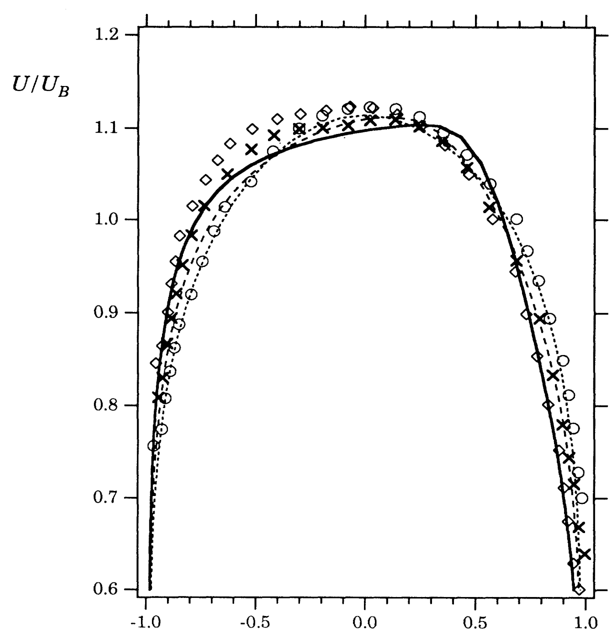

FIG. 1. Mean velocity U. Computation [31]: ——, $R=35\,115$, $R_0=0.2$; — — —, $R=34\,962$, $R_0=0.068$; · · · ·, $R=35\,009$, $R_0=0$. Experiment [32]: ◇, $R=35\,000$, $R_0=0.42$; ×, $R=35\,000$, $R_0=0.068$; ○, $R=36\,000$, $R_0=0$.

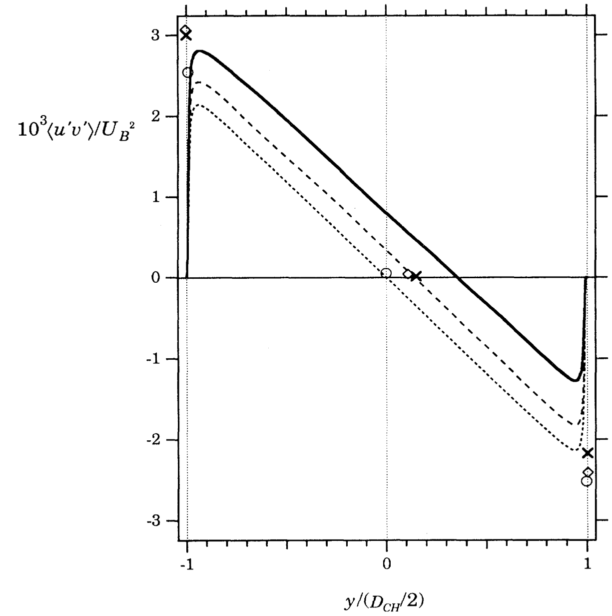

FIG. 2. Shear stress (the symbols are the same as in Fig. 1).

the ϵ equation, we need a more elaborate ϵ equation. Its construction is left for future interesting work.

VI. TRAJECTORY EFFECTS ON THE EDDY VISCOSITY

In Sec. V we, proposed to include the effect of the Lagrange derivative into the isotropic-eddy-viscosity representation $(B_{ij})_E$ [Eq. (51)]. This effect expresses the convection effect on K and ϵ through $(D/Dt)\nu_{E,S}$. In what follows, we shall call it the trajectory effect and discuss its implication in the context of effects of an adverse pressure gradient.

Besides the shortcomings referred to in Sec. V, the current isotropic-eddy-viscosity representation suffers from the deficiency related to an adverse pressure gradient. Under such a gradient, the streamwise velocity U is retarded and the flow often separates from a solid wall. The effect of an adverse pressure gradient most typically appears through negative $\partial U/\partial x$ (x is the streamwise coordinate and y is the coordinate normal to it). Under the eddy-viscosity representation for the Reynolds stress, however, the K production rate P_K in a turbulent boundary layer is insensitive to the streamwise change of U, since P_K is approximated as $R_{xy}(\partial U/\partial y) \simeq \nu_{E,S}(\partial U/\partial y)^2$ and does not depend directly on $\partial U/\partial x$ [$\nu_{E,S}$ is given by Eq. (23)]. This point was discussed in detail by Rodi and Sheuerer [34]. As a method for incorporating the effect of $\partial U/\partial x$, Hanjalic and Launder [35] proposed to include the effect of irrotational strain in the ϵ equation. Another method is to include the trajectory effect given by D/Dt, which also expresses a kind of streamwise-change effect. In what follows, we shall adopt the latter viewpoint.

A prominent feature of an adverse pressure gradient is that the shear stress R_{xy} increases greatly, whereas the mean velocity gradient $\partial U/\partial y$ does not change as much. As a result, the eddy viscosity $\nu_{E,S}$ $(=C_{E,S}K^2/\epsilon)$ increases. The difficulty with the current eddy-viscosity models under an adverse pressure gradient lies in the overestimate of $\nu_{E,S}$. The entirely similar situation is also encountered in a turbulent flow past a bluff body (its typical example is a flow past a rectangular building). In this case, the flow in the front side of the body is retarded and the turbulent intensities increase. The models of K-ϵ type based on the eddy viscosity overestimate them, compared with the observational and large eddy simulation results. This point was discussed in detail by Murakami, Mochida, and Hayashi [36,37].

A feature of the above-stated flows lies in the streamwise increase in the eddy viscosity $\nu_{E,S}$ under the effect of the retardation of the streamwise velocity. In this case, we have positive $(D/Dt)\nu_{E,S}$. In the present model (51), $\nu_{E,S}$ is multiplied by the factor $[1+C_{TE}(K/\epsilon)(D/Dt)\ln\nu_{E,S}]^{-1}$. As a result, this factor bears a role in alleviating the excessive increase in the net eddy viscosity ν_E under the adverse pressure gradient. The attempt to include effects of streamwise change was also proposed for the subgrid-scale modeling [38], and the improvement of the Smagorinsky model was confirmed [39].

VII. CONCLUDING REMARKS

In this work, we made full use of the TSDIA results for the Reynolds stress to bridge a gap between the eddy-viscosity-type and second-order models that have been studied rather independently. Through this work, we added some effects such as the turbulent-helicity one into the conventional second-order models. Moreover a higher-order eddy-viscosity-type expression for the Reynolds stress, which is beyond the analysis of the TSDIA, was obtained by applying an iterative approximation to the second-order model. Specifically, the usefulness of the presently found frame-rotation effect was confirmed in the study of a rotating channel flow. The inclusion of trajectory effects in the isotropic eddy viscosity was also proposed for the study of turbulent flows under adverse pressure gradients.

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